CONSTRUCTION OF ORTHOGONAL SERIES BALANCED INCOMPLETE BLOCK DESIGNS - A NEW TECHNIQUE

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Abstract

The orthogonal series balanced incomplete block designs with parameters $v = n^2$, $b = n(n+1)$, $r = n+1$, $k = n$, and $\lambda = 1$ have been constructed by the use of permutation matrices for n a prime power. The construction problem arose in connection with a query from the Educational Testing Service, Princeton, New Jersey, regarding the existence of such designs.

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* On leave from Punjab Agricultural University, India.

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BALANCED INCOMPLETE BLOCK DESIGNS -
A NEW TECHNIQUE *

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1. Introduction

The orthogonal series (OS 1) balanced incomplete block (BIB) designs with parameters

$$v = n^2,\ b = n(n+1),\ r = n+1,\ k = n,\ \lambda = 1 \qquad (1.1)$$

are known to exist when n is a prime or a prime power, and the construction is based on finite geometries or complete sets of mutually orthogonal latin squares (cf. Raghavarao (1971)). However, this series can also be constructed in an interesting way by using permutation matrices when n is a prime or a power of a prime, and we describe this method in this communication.

2. Permutation matrices as a group

Let $n = p^s$, where p is a prime and s is a positive integer and let $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ be the elements of $GF(p^s)$. Let $A^* = (\alpha_i + \alpha_j)$ be an $n \times n$ matrix representing the addition table of the field elements and let A be obtained from A^* by row permutations such that α_0 is in all the diagonal positions. We define n

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matrices $P_0, P_1, \ldots, P_{n-1}$, where P_i is formed by putting a one in the locations in P_i where α_i occurs in A $(i = 0,1,\ldots,n-1)$. Clearly $P_0 = I_n$, and

$$\sum_{i=0}^{n-1} P_i = J_n ,$$

$$P_i P_i' = P_i' P_i = I_n , \quad i = 0,1,\ldots,n-1 \tag{2.1}$$

where J_n is an nxn matrix with one in all the n^2 positions

In the set $P = \{P_0, P_1, \ldots, P_{n-1}\}$ let * denote the ordinary matrix multiplication. We now have the following:

<u>Theorem 2.1</u>: <u>The system</u> $\tilde{P} = \{P, *\}$ <u>is a group</u>.

<u>Proof</u>: The operation * is closed in P. For, let $P_t, P_u \in P$. In the ith row of $P_t * P_u$, we will have one if and only if

$$\begin{aligned} \alpha_i + \alpha_k &= \alpha_t , \\ \alpha_\ell + \alpha_j &= \alpha_u , \end{aligned} \tag{2.2}$$

where $\alpha_k + \alpha_\ell = \alpha_0$. Thus $\alpha_i + \alpha_j = \alpha_t + \alpha_u$ = Const. and this uniquely determines j by j_i and $P_t * P_u$ will have one's in (i, j_i) positions for $i = 0,1,\ldots,n-1$. If $\alpha_t + \alpha_u = \alpha_w$, then $P_t * P_u = P_w$. Since matrix multiplication is associative, the associativity law holds. P_0 is the group identity and the inverse of P_i is P_j if $\alpha_i + \alpha_j = \alpha_0$. This completes the proof of the theorem.

We also note that if $\alpha_i + \alpha_j = \alpha_0$, then $P_j = P_i'$, where P_i' denotes the transpose of P_i.

In the proof of the above theorem, we have also established the following theorem.

Theorem 2.2: The mapping $F \xrightarrow{f} P$ given by $f(\alpha_i) = P_i$ for $i = 0,1,\ldots,n-1$ is an isomorphism of the groups. $\tilde{F} = \{F, +\}$ and $\tilde{P} = \{P, *\}$, where $F = \{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\}$.

3. Main Result

Let $M^* = (\alpha_i\alpha_j)$ be an nxn matrix denoting the multiplication table of the field elements and let M be a $n^2 \times n^2$ matrix obtained from M^* by replacing α_i's by P_i matrices $(i = 0,1,\ldots,n-1)$. Let T_i be an nxn matrix with one's in the ith column $(i = 0,1,\ldots,n-1)$. Then we have the following main result.

Theorem 3.1: The $n^2 \times n(n+1)$ matrix

$$N = \begin{bmatrix} T_0 & \\ T_1 & \\ \vdots & M \\ T_{n-1} & \end{bmatrix} \tag{3.1}$$

represents the incidence matrix of the OS 1 BIB design with parameters (1.1).

The proof of the theorem follows by verifying that

$$NN' = nI_n + J_n \tag{3.2}$$

after noting that each of the field elements is represented exactly once among the differences of the corresponding elements in the ith and jth rows of M^* for $i,j = 0,1,\ldots,n-1$, $i \neq j$ and making use of Theorem 2.2.

4. An Example

Let us form the incidence matrix N for the design with parameters (1.1) when $n = 4$. The field elements are $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = x$, $\alpha_3 = x^2$ where $x^2 + x + 1 = 0$, and

$$A^* = A = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} . \tag{4.1}$$

Now, we introduce

$$P_0 = I_4,\ P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\quad P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{4.2}$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} . \qquad (4.2 \text{ cont'd.})$$

Again, the multiplication table of the field elements is

$$M^* = \begin{bmatrix} \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_0 & \alpha_2 & \alpha_3 & \alpha_1 \\ \alpha_0 & \alpha_3 & \alpha_1 & \alpha_2 \end{bmatrix} . \qquad (4.3)$$

Let

$$T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} , \quad T_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(4.4)

$$T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \quad T_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Now

$$N = \begin{bmatrix} T_0 & P_0 & P_0 & P_0 & P_0 \\ T_1 & P_0 & P_1 & P_2 & P_3 \\ T_2 & P_0 & P_2 & P_3 & P_1 \\ T_3 & P_0 & P_3 & P_1 & P_2 \end{bmatrix} \tag{4.5}$$

is the required incidence matrix.

5. Concluding Remarks

The main result was obtained by Federer and Raghavarao [1972] when n is a prime number. This result was extended by Joiner and Federer [1973] by using group automorphisms techniques. However, for obtaining the main result of this paper, the introduction of permutation matrices as made in the text is sufficient.

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